A THEOREM ON EXHAUSTIBLE SETS CONNECTED WITH DEVELOPMENTS OF POSITIVE REAL NUMBERS.*

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1. We start with any *aggregate* A whatsoever, and deal with "*developments*" of positive real numbers ξ in the form of an infinite sequence

$$(D) \qquad \xi = \{x_1, x_2, \cdots x_n, \cdots\},$$

where x_n belongs to A. By a "*development of* ξ," we here understand simply an infinite sequence of elements of A, *associated* with ξ according to any given law, as yet—but not eventually—unrestricted. The equality sign in (D) signifies nothing more than that the infinite sequence is *associated* with ξ. We assume at once, as the term "development" might indicate, that *two different* ξ's *cannot have identical developments* (D). We do *not* demand that every ξ shall possess at least one development, nor that the development of ξ shall be unique. However, we do suppose that *no* ξ *has more than a finite number of different developments* (D).

If ξ possesses at least one development (D), we call it "*developable*"; otherwise, "*non-developable.*"

If the infinite sequence $\{x_1, x_2, \cdots x_n, \cdots\}$ is a development of a real number ξ, we say that it is a "*proper development*"; if no ξ exists that has the sequence as a development, we call it an "*improper development.*" In the former case, ξ is called the "*prototype*" of $\{x_1, x_2, \cdots x_n, \cdots\}$.

According to the above, the prototype of a proper development is unique; and the development of a developable number may be multiply but not infinitely "valued."

In the case of the ordinary infinite decimal development, A consists of the set $(0, 1, \cdots 9)$. Here every ξ is developable, but rational numbers may have two developments. A will consist of the set of positive integers, if we represent $\xi(< 1)$ as an infinite continued fraction in the form

$$\xi = \cfrac{1}{x_1 + \cfrac{1}{x_2 +}}$$

* Read before the American Mathematical Society, December, 1913. We use (after Denjoy) the term "*exhaustible*"—instead of the phrase, "of first category" (Baire)—to denote a set that is the sum of a countable set of non-dense sets; and later,—see Theorem below—the term "*residual,*" to denote the complementary set of an exhaustible set. Cf. Denjoy, Journal de Mathématiques, ser. 7, vol. 1 (1915), pp. 122–125. Instead of restricting ourselves to positive real numbers, we might discuss all real numbers; but the gain would be negligible.

requiring x_n to be a positive integer. In this case, rational numbers are non-developable.

For our purpose, we demand that the development (D) shall possess the

Property P. If in (D), the elements $x_1, x_2, \cdots x_n$ are fixed, and the remaining elements x_ν, $\nu > n$ range in every possible manner over the aggregate A, then the set of prototypes of the proper developments thus obtained constitute the totality of developable numbers in one and the same interval—closed or open on one or both sides—of the linear continuum.*

It is to be understood that such an interval is allowed to extend to ∞, if necessary. We distinctly do not demand the unique existence of such an interval; but there is always a definite "largest,"† which may be obtained from any interval of the described character, by extending the latter as far as possible both to the left and the right, with the restriction that no interior developable numbers shall be allowed that do not belong to the set of prototypes obtained in the definition of P‡. We shall speak of this largest interval as "the interval associated with the sequence $(x_1, x_2, \cdots x_n)$."

Property P is evidently possessed by the ordinary decimal development and by the continued fraction development described above.

The object of the present note is to communicate the following:

THEOREM. *Let (C) be any condition whatsoever that mates—uniquely or not—with every finite sequence $(x_1, x_2, \cdots x_n)$ of elements of A a finite sequence $(x_1', x_2', \cdots x_p')$ of elements of A. Let S be the totality of developable real numbers ξ possessing no development (D) in which it happens infinitely often that the sequence $(x_1, x_2, \cdots x_n)$ is immediately succeeded by one of its mates, $(x_1', x_2', \cdots x_p')$, so that*

$$x_{n+1} = x_1', \qquad x_{n+2} = x_2', \qquad \cdots x_{n+p} = x_p'.$$

Then S is exhaustible, and the complementary set, consisting of non-developable numbers and of those developable numbers that have at least one development in which it happens for an infinite number of values of n that the

* Our desire here is not that of attaining, as can be done without great difficulty, postulates of extreme simplicity. It is rather that of at once formulating the essential property of the developments (D), that is possessed by the well-known developments and is sufficient for the proof of the Theorem of this note.

† Except possibly when the set of prototypes defined in Property P is the null-set; but this case is trivial for the applications of P.

‡ The largest interval may be defined also as the "sum"—in the sense of the Theory of Aggregates—of all possible intervals of the described character.

*sequence of the first n elements is immediately succeeded by one of its mates, is a residual set.**

Proof. Let S_k represent the set of developable numbers that possess no development (D) in which it happens after $n = k$ that the sequence $(x_1, x_2, \cdots x_n)$ is immediately succeeded by one of its mates according to (C). It follows, by the use of the fact that no number has an infinity of different developments, that every element of S belongs to at least one S_k. To prove that S is exhaustible, we shall prove that S_k is a non-dense set. This will be proved, if we show that in every neighborhood of a given developable number ξ, with the development (D), there is an interval free from points of S_k. If ξ is not a limit from both sides of developable numbers, the existence of such an interval is obvious. We may and do assume, therefore, without loss of generality, that ξ is a both-sided limit of developable numbers. With every sequence $(x_1, x_2, \cdots x_n)$, there is associated, according to Property P, a definite interval I_n, containing ξ. As n approaches ∞, the length m_n of I_n must approach zero. For

$$I_1 \supseteqq I_2 \supseteqq I_3 \supseteqq \cdots;$$

and if m_n does not approach zero, I_n approaches a definite interval I—closed or open on one or both sides—containing ξ. As ξ is a both-sided limit of developable numbers, I must contain a developable number $\xi' \neq \xi$. This number, lying in I_n, has as the first n elements of one of its developments the sequence $(x_1, x_2, \cdots x_n)$. Since this holds for every n, and ξ' has no more than a finite number of developments, it follows that at least one development of ξ' is identical with the development (D) of ξ, contrary to the uniqueness of the prototype of a proper development. Consequently,

$$\lim_{n \to \infty} m_n = 0.$$

Let now $n > k$, and let the sequence $(x_1', x_2', \cdots x_p')$ be a mate of the sequence $(x_1, x_2, \cdots x_n)$ according to (C). The interval I_n' associated with the sequence $(x_1, x_2, \cdots x_n, x_1', x_2', \cdots x_p')$ contains no element of S_k, since I_n' contains only non-developable numbers or developable numbers having at least one development beginning with the sequence $(x_1, x_2, \cdots x_n, x_1', x_2', \cdots x_p')$. Moreover, I_n' is contained in the interval I_n associated with the sequence $(x_1, x_2, \cdots x_n)$. Since, by taking n sufficiently large, the interval I_n may, according to $\lim m_n = 0$, be

* Condition (C) may, in the terminology of E. H. Moore—see his *Introduction to a Form of General Analysis* (1910), preface—be described as a function on M to M, where M stands for the set of finite sequences of elements of A. These finite sequences enter fundamentally in the writer's paper, On the Factorization of Expressions of Various Types, Trans. of the Am. Math. Soc., vol. 17 (1916), pp. 517–544, where they are called "parentheses."

brought within any given neighborhood, however small, of ξ, it follows that the interval I_n', which is free from elements of S_k, may be likewise brought within such a neighborhood. The non-density of S_k is thus established.

Since the decimal development and the continued fraction development described above are special cases of the general development (D), we have the following consequences, formulated, for the sake of simplicity, for the set of numbers between 0 and 1:

COROLLARY 1. *Let (C) be any condition that mates with every finite sequence $(x_1, x_2, \cdots x_n)$, where x_ν is one of the integers* 0, 1, 2, $\cdots$ 9, *at least one sequence of the same character. Then the set of positive real numbers < 1, in whose decimal development the sequence $(x_1, x_2, \cdots x_n)$ is immediately followed by one of its mates for only a finite number of values of n, is exhaustible.*

COROLLARY 2. *Let the positive real number ξ (< 1) be represented as an infinite continued fraction in the form*

$$\cfrac{1}{x_1 + \cfrac{1}{x_2 + \cdots}},$$

the x's being positive integers. Let (C) be any condition that mates with every finite sequence of positive integers at least one sequence of the same character. Then the set of positive real numbers < 1, in whose continued fraction development defined above, the sequence $(x_1, x_2, \cdots x_n)$ is immediately followed by one of its mates for only a finite number of values of n, is exhaustible.

There is no difficulty in seeing that *the Theorem applies to all the well-known developments of real numbers.* In addition to those already referred to, we mention the decimal development in the scale of k, the generalized decimal development of Strauss,* and the development as an infinite product, due to Cantor.†

For purposes of illustration, we mention several particular cases of Corollary 1.

(*a*) Let (C) mate with every finite sequence the integer 1. Then, according to Corollary 1, *the set of decimal fractions in which* 1 *occurs only finitely often is exhaustible.*

(*b*) Let (C) mate with the sequence $(x_1, x_2, \cdots x_{n-1})$ a sequence of $f_n + 2$ terms, the first and the last being $\neq 1$, and the f_n others $= 1$; here f_n stands for a given function of n taking only positive integral values.

* Acta Math., vol. 11 (1887–88), pp. 13–18.

† See Hobson, "The Theory of Functions of a Real Variable" (1907), p. 48.

Then according to Corollary 1, *the totality of decimal fractions in which it happens infinitely often that an unbroken sequence of* 1's *commences exactly at the* $(n + 1)$*th place and ends exactly at the* $(n + f_n)$*th place, is a residual set.*

(*c*) Let n_1 be the number of 1's in the first n figures of the decimal development of ξ. Then as follows readily from (*b*) or Corollary 1, *the totality of the numbers* ξ *for which* n_1/n *has, as* n *approaches* ∞, *every number between* 0 *and* 1 *as a limit, is a residual set.*

2. Remarks on generalizations and connection with certain results of Hardy and Littlewood. (*a*) The use of the term "immediately" in the clause, "that the sequence $(x_1, x_2, \cdots x_n)$ is immediately succeeded by one of its mates," involves less of a restriction than may appear at first sight. For example, *there is no gain whatsoever in generality in replacing "immediately succeeded by one of its mates . . ." by "succeeded by a mate* $(x_1', x_2', \cdots x_p')$ *that begins exactly at the* $(n + f_n + 1)$*th place of the development, so that*

$$x_{n+f_n+1} = x_1', \qquad x_{n+f_n+2} = x_2', \qquad \cdots,\text{"}$$

where f_n stands for a given function of n independent of condition (C) and taking only positive integral values. For, by using instead of the given condition (C), one that mates with $(x_1, x_2, \cdots x_n)$ every sequence $(x_{n+1}, x_{n+2}, \cdots x_{n+f_n}, x_1', x_2', \cdots x_p')$, where $x_{n+1}, x_{n+2}, \cdots x_{n+f_n}$ take independently all possible values, we are led back precisely to the situation of the Theorem. Again, it may be shown in like manner, that there would be no gain in substituting for the sequence $(x_1, x_2, \cdots x_n)$ the sequence $(x_{\lambda_n}, x_{\lambda_n+1}, \cdots x_{\mu_n})$, where λ_n and μ_n are given functions of the character of f_n described above; nor is anything gained in employing simultaneously both apparent generalizations just described.

(*b*) The supposition that no ξ possesses more than a finite number of developments may be replaced by the following weaker condition: If $\{x_1, x_2, \cdots x_n, \cdots\}$ is a given proper development, and if ξ has, for every value of n, at least one development beginning with the sequence $(x_1, x_2, \cdots x_n)$ then ξ has the given development as one of its developments. However, if this modification is introduced, the wording of the Theorem should be changed to read "possessing no development (D) in which it happens more than k_ξ times . . ." instead of "possessing . . . infinitely often," where k_ξ is an integer depending on ξ, but not on its various developments; and the latter part of the Theorem should, of course, be correspondingly modified.

(*c*) The generality of condition (C)—see also the first remark—is such that one is tempted to say, in a loose description of the Theorem, denoting the sequence $(x_1, x_2, \cdots x_n)$ as an "approximate development of

ξ," that "the set of those real numbers that have only a finite number of approximate developments in which a given thing happens, is almost always exhaustible." If we conceive, as has been suggested, of an exhaustible set as one "qualitatively poor" in elements, and a set of measure zero as one "quantitatively poor," the Theorem offers an unexpected contrast with results of Hardy and Littlewood on the measure of certain point sets.* For example, in contrast even to the particular case (*c*) of Corollary 1, one of the theorems of Hardy and Littlewood states† that *the totality of real numbers between* 0 *and* 1 *for which* n_1/n, *as* n *approaches* ∞, *has the limit* $1/10$ *is of measure* 1.

(*d*) The extension of our Theorem to more general sets than that of the real numbers—in particular, to the set of points in n-space—may be made with only slight modification of our assumptions; but we shall not enter here into the discussion of such extensions.

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* Acta Mathematica, vol. 37 (1914), pp. 155–190, especially p. 183 et seq.

† Loc. cit., p. 186.